Complementarity in the Study of Transmission Lines*

G. H. OWYANG† AND RONOLD KING‡

Summary—The principle of complementarity is applied to the slot transmission line. The properties of a dual circuit are investigated. The pairs of several possible duals for a given configuration are correlated and new quantities are defined for use with different types of circuits. A complete parallelism between the two-wire line and the two-slot line is established for the ideal cases and is extended by approximation to include the practical cases.

Measurements were made with a two-slot transmission line and its associated probing system. The method of testing the line for balance is discussed. The transverse distribution of the longitudinal current and the attenuation constant were measured.

The analogy between the steady-state field in a conducting medium and the electrostatic field in a dielectric is investigated. The expressions for the constants of a two-slot line are given in a form that permits a ready evaluation from experimental data obtained with the electrolytic tank. The measured results are compared with theoretical values.

I. The Principle of Complementarity

A. Introduction

If two physically different phenomena, A and B, are described by the same mathematical formulation, quantitative conclusions may be drawn about A from a study of B. This is true of complementary problems in electromagnetic theory, in which the field about a configuration A of slots in a perfectly conducting infinite plane of zero thickness is related to the field about a configuration B of conducting strips arranged in free space to correspond exactly to the slots in A.

B. Duality Between the Electromagnetic Field of an Electric and a Magnetic Source

Consider groups of perfect electric and perfect magnetic conductors in a homogeneous medium characterized by the complex permittivity $\epsilon=\epsilon_e-j\sigma_e/\omega$, and the permeability μ (see Fig. 1). S_1, S_2, $\cdots$ are the surfaces of the electric conductors, S_1^*, S_2^*, $\cdots$ of the magnetic conductors. The appropriately generalized field and continuity equations are

$$\text{curl}\, \boldsymbol{E} = -\boldsymbol{J}^* - j\omega\mu \boldsymbol{H}, \qquad \text{div}\, \boldsymbol{E} = \rho/\epsilon; \tag{1a}$$

$$\text{curl}\, \boldsymbol{H} = \boldsymbol{J} + j\omega\epsilon \boldsymbol{E}, \qquad \text{div}\, \boldsymbol{H} = \rho^*/\mu; \tag{1b}$$

$$\text{div}\, \boldsymbol{J} + j\omega\rho = 0 \qquad \text{div}\, \boldsymbol{J}^* + j\omega\rho^* = 0. \tag{1c}$$

(The symbols are defined in Fig. 4.) The boundary conditions on the surfaces of the conductors are

$$\hat{n} \times \boldsymbol{E} = \hat{n}\cdot \boldsymbol{H} = 0 \tag{2a}$$

on the electric conductors S_1, S_2, $\cdots$, and

$$\hat{n} \times \boldsymbol{H} = \hat{n}\cdot \boldsymbol{E} = 0 \tag{2b}$$

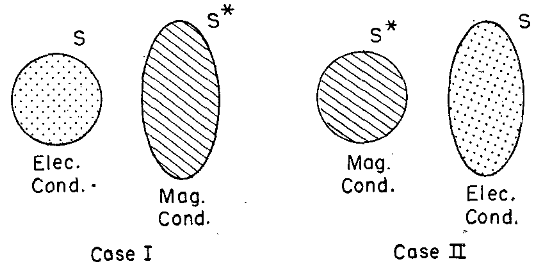


Fig. 1

on the magnetic conductors S_1^*, S_2^*, $\cdots$, where n is a unit outward normal.

It can be shown that an interchange of the electric and magnetic sources and conductors in a given system results in an interchange of the E- and H-fields. In particular, if

$$J_2 = -\eta_e J_1^*, \qquad J_2^* = \zeta_e J_1, \tag{3a}$$

$$\rho_2 = -\eta_e \rho_1, \qquad \rho_2^* = \zeta_e \rho_1, \tag{3b}$$

where $\zeta_e^2=1/\eta_e^2=\mu/\epsilon$, the field vectors are given by

$$\boldsymbol{E}_2 = -\zeta_e \boldsymbol{H}_1, \qquad \boldsymbol{H}_2 = \eta_e \boldsymbol{E}_1. \tag{4}$$

The subscripts 1 and 2 refer to cases I and II (Fig. 1), respectively.

C. Fields with E-symmetry and H-symmetry.

In rectangular coordinates the field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are E-symmetric (or H-antisymmetric) with respect to the plane $x=0$ if

$$E_u(x) = \begin{cases} -E_u(-x), & u = x \\ E_u(-x), & u = y \text{ or } z \end{cases}$$

$$H_u(x) = \begin{cases} H_u(-x), & u = x \\ -H_u(-x), & u = y \text{ or } z. \end{cases} \tag{5}$$

The shorthand notations $F(x)$ and $F(-x)$ are used for $F(x, y, z)$ and $F(-x, y, z)$. The corresponding field vectors with H-symmetry (or E-antisymmetry) are

$$E_u(x) = \begin{cases} E_u(-x), & u = x \\ -E_u(-x), & u = y \text{ or } z, \end{cases}$$

$$H_u(x) = \begin{cases} -H_u(-x), & u = x \\ H_u(-x), & u = y \text{ or } z. \end{cases} \tag{6}$$

With these definitions, any function $\boldsymbol{F}(x)$ may be expressed as the sum of symmetric and antisymmetric components in the form $\boldsymbol{F}(x)=\boldsymbol{F}_s(x)+\boldsymbol{F}_a(x)$ where

$$\boldsymbol{F}_i(x) = \tfrac{1}{2}\{\hat{x}[F_x(x) \mp F_x(-x)] + \hat{y}[F_y(x) \pm F_y(-x)] + \hat{z}[F_z(x) \pm F_z(-x)]\}. \tag{7}$$

$i=s$ for the upper signs, $i=a$ for the lower signs.

* Manuscript received by the PGMTT, June 4, 1959; revised manuscript received, October 20, 1959. This research was supported jointly by the Navy Department (ONR), the Signal Corps of the U. S. Army, and the U. S. Air Force, under contract Nonr. 1866(32.)

† Rad. Lab., University of Michigan, Ann Arbor, Mich. Formerly at Gordon McKay Lab., Harvard University, Cambridge, Mass.
‡ Gordon McKay Lab., Harvard University, Cambridge, Mass.

With (5)–(7) it follows directly that for a structure in space that is symmetric, the field equations are independent of the sign of x and may be separated into E-symmetric and H-symmetric parts. These are, for H-symmetry,

$$\text{curl } \boldsymbol{E}_a(x) + j\omega\mu \boldsymbol{H}_s(x) = -\boldsymbol{J}_s^*(x), \tag{8a}$$

$$\text{curl } \boldsymbol{H}_s(x) - j\omega\epsilon \boldsymbol{E}_a(x) = \boldsymbol{J}_a^*(x); \tag{8b}$$

for E-symmetry,

$$\text{curl } \boldsymbol{E}_s(x) + j\omega\mu \boldsymbol{H}_a(x) = -\boldsymbol{J}_a^*(x), \tag{8c}$$

$$\text{curl } \boldsymbol{H}_a(x) - j\omega\epsilon \boldsymbol{E}_s(x) = \boldsymbol{J}_s(x). \tag{8d}$$

It follows from (5) that at $x=0$,

$$E_{sx}(0) = H_{ay}(0) = H_{az}(0) = 0 \tag{9}$$

so that in a homogeneous medium the E-symmetric field satisfies the boundary conditions (2) of a perfect magnetic conductor at $x=0$, and is not disturbed by the insertion of a plane sheet of perfect magnetic conductor of arbitrary shape and size in the plane of symmetry. Similarly, from (6)

$$E_{ay}(0) = E_{az}(0) = H_{sx}(0) = 0, \tag{10}$$

so that the H-symmetric field satisfies the boundary conditions (2) of a perfect electric conductor at $x=0$ and is undisturbed by the insertion of a plane sheet of perfect electric conductor in the plane of symmetry.

D. Duality Between a Thin Disk and a Hole in a Thin Sheet

Let a thin disk made of a perfect electric conductor be placed in the plane of symmetry ($x=0$) in a homogeneous medium as shown in Fig. 2(a). In this medium there exists an electromagnetic field maintained by a symmetric distribution of electric currents $\boldsymbol{J}_s(x)$. It follows from (8) that the field is E-symmetric so that it behaves just as if a magnetic conductor were located in the plane of symmetry outside the disk as shown in Fig. 2(b).

If the electric and magnetic conductors are interchanged, Fig. 2(d) is obtained. Since the regions $x>0$ and $x<0$ are separated by the sheets of conductor, (3) may be used with opposite signs in these two regions. That is, the distribution of magnetic current is antisymmetric for the new system, so that it satisfies the following relations:

$$\boldsymbol{J}_{a4}^*(x) = \zeta_e \boldsymbol{J}_{s3}(x), \quad \boldsymbol{J}_{a4}^*(-x) = -\zeta_e \boldsymbol{J}_{s3}(-x), \tag{11}$$

where the subscripts 3 and 4 refer to the systems before and after the change. Since the excitation is by antisymmetric magnetic currents, the field has E-symmetry and it is immaterial whether the sheet of magnetic conductor is present or not. Therefore, Figs. 2(c) and 2(d) represent equivalent configurations and the formulas shown in the figure follow directly from (3) and (4).

The following situations have been shown to be duals: a thin disk of perfect electric conductor in the plane of symmetry of an E-symmetric field that is excited by a symmetrical distribution of electric currents; an infinite sheet of electric conductor with a hole that has the same size and shape as the disk if the sheet is placed in the plane of symmetry of an E-symmetric field that is excited by an antisymmetric distribution of magnetic currents.

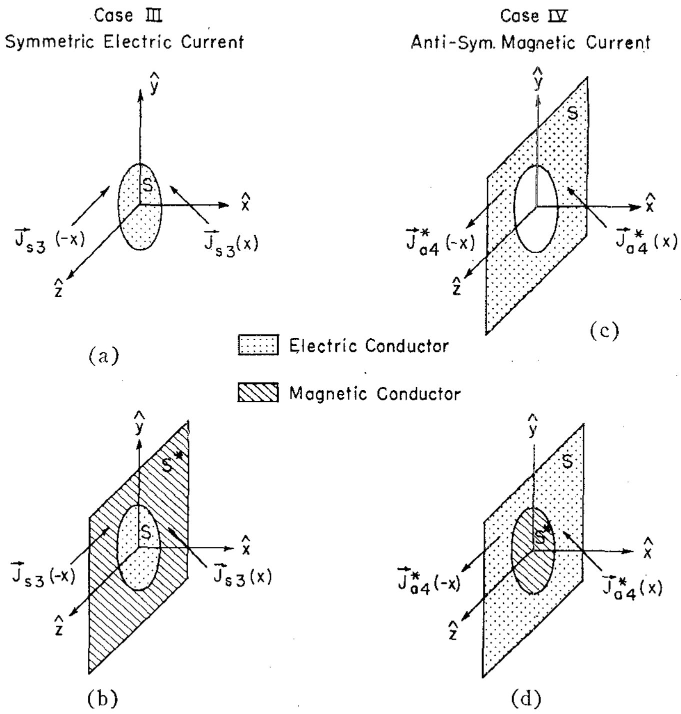


Fig. 2—Duality between metallic disk and hole in metallic screen.

Field: E-symmetry with respect to $x=0$ plane

$$E_x(x) = -E_x(-x) \qquad H_x(x) = H_x(-x)$$
$$E_y(x) = E_y(-x) \qquad H_y(x) = -H_y(-x)$$
$$E_z(x) = E_z(-x) \qquad H_z(x) = -H_z(-x)$$

$x>0$ | $x<0$

$$\boldsymbol{J}_{a4}^*(x) = \zeta_e \boldsymbol{J}_{s3}(x) \quad v/m^2 \qquad \boldsymbol{J}_{a4}^*(x) = -\zeta_e \boldsymbol{J}_{s3}(x) \quad v/m^2$$
$$\boldsymbol{E}_4(x) = -\zeta_e \boldsymbol{H}_3(x) \quad v/m \qquad \boldsymbol{E}_4(x) = \zeta_e \boldsymbol{H}_3(x) \quad v/m^a$$
$$\boldsymbol{H}_4(x) = \eta_e \boldsymbol{E}_3(x) \quad a/m \qquad \boldsymbol{H}_4(x) = -\eta_e \boldsymbol{E}_3(x) \quad a/m$$

$$\eta_e^2 = \frac{1}{\zeta_e^2} = \frac{\epsilon}{\eta^2} = \frac{1}{\mu}\left(\epsilon_e + \frac{\sigma_e}{j\omega}\right) \text{ mho}^2$$

In a similar manner, it can be shown that a magnetic conducting disk in an electromagnetic field that is excited by symmetric magnetic currents is the dual of a similar hole in a magnetic conducting sheet located in a field that is generated by an antisymmetric distribution of electric currents. The arrangements for these two cases are shown in Fig. 3.

E. Terminology

Since several dual configurations may be defined for a given structure, it is desirable to label each type of network unambiguously. The circuit made of ordinary electrically conducting strips is the *actual electric circuit* or the *electric strip circuit*, the complementary circuit made of fictitious magnetic strips is the *fictitious magnetic circuit* or the *magnetic strip circuit*, and the complementary circuit obtained by cutting slots in a metallic

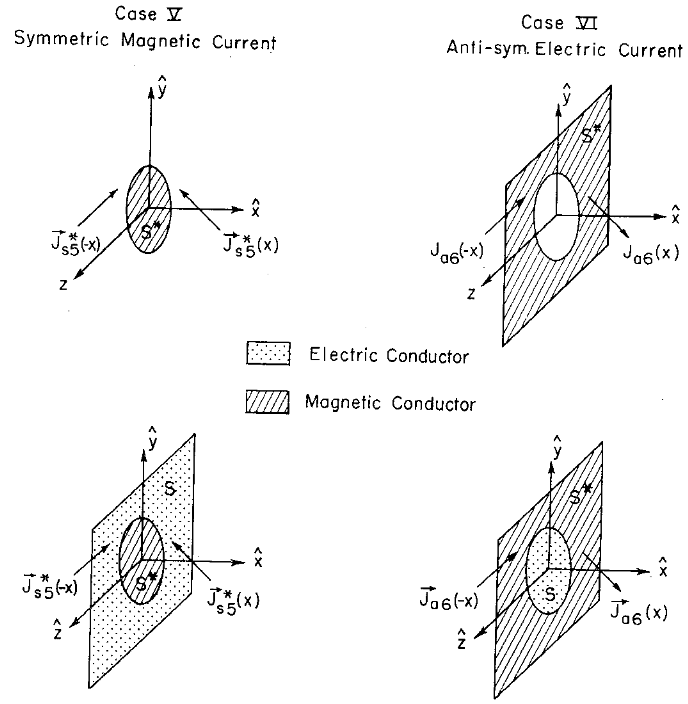


Fig. 3—Duality between magnetic disk and hole in magnetic screen.

Field: H-symmetry with respect to $x=0$ plane

$$E_x(x) = E_x(-x) \qquad H_x(x) = -H_x(-x)$$
$$E_y(x) = -E_y(-x) \qquad H_y(x) = H_y(-x)$$
$$E_z(x) = -E_z(-x) \qquad H_z(x) = H_z(-x)$$

$$x>0 \quad J_{a6}(x) = -\eta_e J_{s5}{}^*(x) \ a/m^2 \qquad x<0 \quad J_{a6}(x) = \eta_e J_{s5}{}^*(x) \ a/m^2$$
$$E_6(x) = -\zeta_e H_5(x) \ v/m \qquad E_6(x) = \zeta_e H_5(x) \ v/m$$
$$H_6(x) = \eta_e E_5(x) \ a/m \qquad H_6(x) = -\eta_e E_5(x) \ a/m$$
$$\eta_e^2 = \frac{1}{\zeta_e^2} = \frac{\epsilon}{\mu} = \frac{1}{\mu}\left(\epsilon_e + \frac{\sigma_e}{j\omega}\right) \text{ mho}^2$$

surface is the *slot circuit*. The dual obtained by replacing a given original configuration of conductors by its complement is called the *physical dual*. For example, a metallic disk is the physical dual of a hole of similar shape in a metallic screen and vice versa. A system of electric conductors and a similar system of magnetic conductors are *ideal* or *fictitious duals*. A magnetic strip is the ideal dual of a geometrically identical electric strip. A new set of quantities is needed for use in fictitious duals. These are given conventional names preceded by the world "magnetic." An asterisk is attached to the symbol for such a magnetic quantity for identification, as shown in Fig. 4.

The quantities used to describe a slot circuit are preceded by the word "complementary" and their symbols are primed to distinguish them from those for electric circuits. The complementary currents and charges are, of course, those maintained on the complementary conducting surfaces. Complementary quantities are listed in Fig. 4.

The duality between pairs of circuits and associated equations is illustrated in Fig. 4. Note that the quantities listed for the electric and magnetic strip circuits are duals. Corresponding quantities for the slot and the magnetic strip circuits are not duals, but some of them are equivalent as indicated in parentheses.

F. Generalized Two-Slot Transmission-Line Theory

An ideal two-slot transmission line consists of two parallel slots that are cut in an infinitely thin, perfectly conducting sheet of infinite size (see Fig. 5). The ideal (although physically fictitious) dual consists of two parallel thin strips, made of a perfect magnetic conductor, that lie in the xy-plane, symmetrically located with respect to the x-axis and with their centers separated by a distance b. If the width a of the strip satisfies the inequalities, $\beta_0 a \ll 1$, $b^2 \gg a^2$, it is proper to define a total axial magnetic current and a total magnetic charge per unit length and to assume that their transverse distributions are approximately symmetrical with respect to the center of each strip. In order to make radiation negligible, the condition $(\beta_0 b)^2 \ll 1$ is imposed.

At distances from both ends of the transmission line that are large compared with the separation b of the strips, the following relations[1] are obtained for the magnetic scalar and vector potential differences $V^*(w)$ and $W_x{}^*(w)$. The same formulas apply to the electric potentials if the asterisks are omitted.

$$\frac{\partial^2}{\partial w^2} V^*(w) - \gamma_0{}^{*2} V^*(w) = 0, \tag{12a}$$

$$\frac{\partial^2}{\partial w^2} W_x{}^*(w) - \gamma_0{}^{*2} W_x{}^*(w) = 0, \tag{12b}$$

$$I_x{}^*(w) = \frac{1}{z_0{}^*} \frac{\partial}{\partial w} V^*(w) \tag{13}$$

where $\gamma_0{}^{*2} = y_0{}^* z_0{}^*$. The magnetic line constants (with asterisk) and their electric duals are summarized as follows:

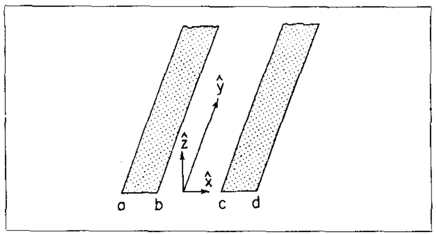
$$z_0{}^* = r_0{}^* + j\omega l_0{}^* = (j\omega\epsilon/\pi) \ln (b/a),$$
$$z_0 = (j\omega\mu/\pi) \ln (b/a); \tag{14a}$$
$$y_0{}^* = j\omega c_0{}^* = (j\omega\mu\pi)/\ln (b/a),$$
$$y_0 = g_0 + j\omega c_0 = (j\omega\pi\epsilon)/ \ln (b/a). \tag{14b}$$

Note that $\epsilon = \epsilon_e - j\sigma_e/\omega$. The magnetic potentials $V^*(w)$ and $W_x{}^*(w)$ for the ideal dual of the two-slot line satisfy the conventional transmission-line equations just as do the potentials $V(w)$ and $W_x(w)$ for the two-wire line. The line constants for the magnetic strips are similar to those for electric strips. The approximate solution for the magnetic current and scalar potential difference may be obtained with a corrective terminal-zone network as for a two-wire line.[2]

Equivalent circuits of the magnetic strip line and the slot line are shown in Fig. 6.

[1] R. W. P. King, "Transmission-Line Theory," McGraw-Hill Book Co., Inc., New York, N. Y., p. 13; 1955.
[2] *Ibid.*, p. 58.

Electric Circuit or Electric Strip Circuit



Original (or Physical Dual)

Field: E-symmetry with respect to $z=0$ plane

E-field	$\boldsymbol{E}$	v/m
H-field	$\boldsymbol{H}$	a/m
Current Density (Volume)	$\boldsymbol{J}$	a/m^2
Current Density (Surface)	$\boldsymbol{K}$	a/m
Charge Density (Volume)	ρ	as/m^3
Charge Density (Surface)	η	as/m^2

Potential Difference

$$V = \int_b^c \boldsymbol{E}\cdot\boldsymbol{dx} \qquad v$$

Current

$$I_y = \int_c^d \hat{n} \times \boldsymbol{H}\cdot\boldsymbol{dx} = -\int_c^d K_y dx \qquad a$$

Impedance

$$z = \frac{V}{I} \qquad \text{ohm}$$

Capacitance per unit length

$$c \qquad \text{f/m}$$

Inductance per unit length

$$l \qquad \text{h/m}$$

Field Equations:	Boundary Conditions:
$\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\epsilon \boldsymbol{E}$	$\hat{n} \times \boldsymbol{H} = -\boldsymbol{K}$
$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}$	$\hat{n} \times \boldsymbol{E} = 0$
$\nabla \cdot \boldsymbol{H} = 0$	$\hat{n} \cdot \boldsymbol{H} = 0$
$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon}\rho$	$\hat{n} \cdot \boldsymbol{E} = \frac{-1}{\epsilon}\eta$

Potential Functions:

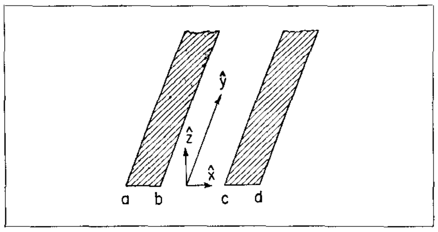
$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A}$$

$$\boldsymbol{A} = \frac{\mu}{4\pi}\int_v \boldsymbol{J} K_0 d\tau'$$

$$\boldsymbol{E} = -\nabla\phi - j\omega \boldsymbol{A}$$

$$\phi = \frac{1}{4\pi\epsilon}\int_v \rho K_0 d\tau'$$

Fictitious Magnetic Circuit or Magnetic Strip Circuit



Ideal or Fictitious Dual

Field: H-symmetry with respect to $z=0$ plane

Magnetic E-field	$\boldsymbol{E}^*(=\boldsymbol{H}')$	a/m
Magnetic H-field	$\boldsymbol{H}^*(=\boldsymbol{E}')$	v/m
Magnetic Current Density (Volume)	$\boldsymbol{J}^*$	v/m^2
Magnetic Current Density (Surface)	$\boldsymbol{K}^*$	v/m
Magnetic Charge Density (Volume)	ρ^*	vs/m^3
Magnetic Charge Density (Surface)	η^*	vs/m^2

Magnetic Potential Difference

$$V^* = \int_b^c \boldsymbol{E}^*\cdot\boldsymbol{dx}(=I') \qquad a$$

Magnetic Current

$$I_y^* = \int_c^d \hat{n} \times \boldsymbol{H}^*\cdot\boldsymbol{dx} = \int_c^d K_y dx(=V') \qquad v$$

Magnetic Impedance

$$z^* = \frac{V^*}{I^*}(=y') \qquad \text{mho}$$

Magnetic Capacitance per unit length

$$c^*(=l') \qquad \text{h/m}$$

Magnetic Inductance per unit length

$$l^*(=c') \qquad \text{f/m}$$

Field Equations:	Boundary Conditions:
$\times \quad = -\boldsymbol{J}^* - j\omega\mu \boldsymbol{E}^*$	$\hat{n} \times \boldsymbol{H}^* = \boldsymbol{K}^*$
$\nabla \times \boldsymbol{E}^* = j\omega\epsilon \boldsymbol{H}^*$	$\hat{n} \times \boldsymbol{E}^* = 0$
$\nabla \cdot \boldsymbol{H}^* = 0$	$\hat{n} \cdot \boldsymbol{H}^* = 0$
$\nabla \cdot \boldsymbol{E}^* = \frac{1}{\mu}\rho^*$	$\hat{n} \cdot \boldsymbol{E}^* = \frac{-1}{\mu}\eta^*$

Magnetic Potential Functions:

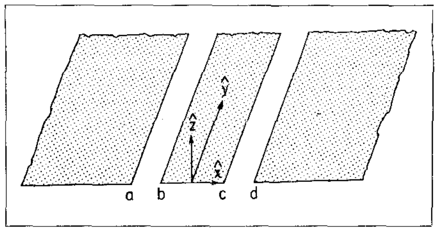
$$\boldsymbol{H}^* = \frac{-1}{\epsilon}\nabla \times \boldsymbol{A}^*$$

$$\boldsymbol{A}^* = \frac{\epsilon}{4\pi}\int_v \boldsymbol{J}^* K_0 d\tau'$$

$$\boldsymbol{E}^* = -\nabla\phi^* - j\omega \boldsymbol{A}^*$$

$$\phi^* = \frac{1}{4\pi\mu}\int_v \rho^* K_0 d\tau'$$

Slot Circuit



Physical Dual (or Original)

Field: E-symmetry with respect to $z=0$ plane

Complementary H-field	$\boldsymbol{H}'(=\boldsymbol{E}^*)$	a/m
Complementary E-field	$\boldsymbol{E}'(=\boldsymbol{H}^*)$	v/m
Complementary Current Density (Volume)	$\boldsymbol{J}'$	a/m^2
Complementary Current Density (Surface)	$\boldsymbol{K}'$	a/m
Complementary Charge Density (Volume)	ρ'	as/m^3
Complementary Charge Density (Surface)	η'	as/m^2

Complementary Current

$$I_y' = \int_b^c \hat{n} \times \boldsymbol{H}'\cdot\boldsymbol{dx} = -\int_b^c K_y dx(=V^*) \qquad a$$

Complementary Potential Difference

$$V' = \int_c^d \boldsymbol{E}'\cdot\boldsymbol{dx}(=I^*) \qquad v$$

Complementary Transverse Admittance

$$y' = \frac{I'}{V'}(=z^*) \qquad \text{mho}$$

Complementary Inductance per unit length

$$l'(=c^*) \qquad \text{h/m}$$

Complementary Transverse Capacitance per unit length

$$c'(=l^*) \qquad \text{f/m}$$

Field Equations:	Boundary Conditions:
$\nabla \times \boldsymbol{H}' = \boldsymbol{J}' + j\omega\epsilon \boldsymbol{E}'$	$\hat{n} \times \boldsymbol{H}' = -\boldsymbol{K}'$
$\nabla \times \boldsymbol{E}' = j\omega\mu \boldsymbol{H}'$	$\hat{n} \times \boldsymbol{E}' = 0$
$\nabla \cdot \boldsymbol{H}' = 0$	$\hat{n} \cdot \boldsymbol{H}' = 0$
$\nabla \cdot \boldsymbol{E}' = \frac{1}{\epsilon}\rho'$	$\hat{n} \cdot \boldsymbol{E}' = \frac{-1}{\epsilon}\eta'$

Complementary Potential Functions:

$$\boldsymbol{H}' = \frac{1}{\mu}\nabla \times \boldsymbol{A}'$$

$$\boldsymbol{A}' = \frac{\mu}{4\pi}\int_v \boldsymbol{J}' K_0 d\tau'$$

$$\boldsymbol{E}' = -\nabla\phi' - j\omega \boldsymbol{A}'$$

$$\phi' = \frac{1}{4\pi\epsilon}\int_v \rho' K_0 d\tau'$$

where $\epsilon = \epsilon_e + \frac{\sigma_e}{j\omega}$, $K_0 = \frac{1}{R}e^{-j\beta_0 R}$, $R^2 = (x-x)^2 + (y-y')^2 + (z-z')^2$

Fig. 4—Complementarity between strip and slot circuits.

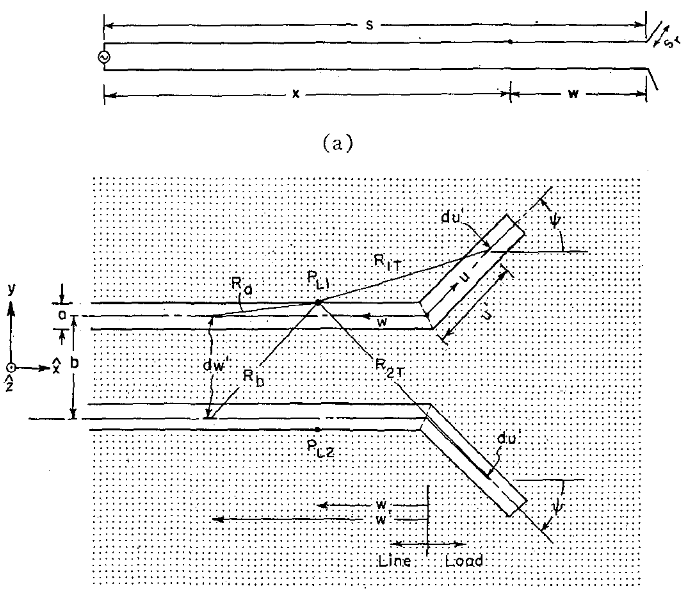


Fig. 5—Arrangement of a two-slot line.

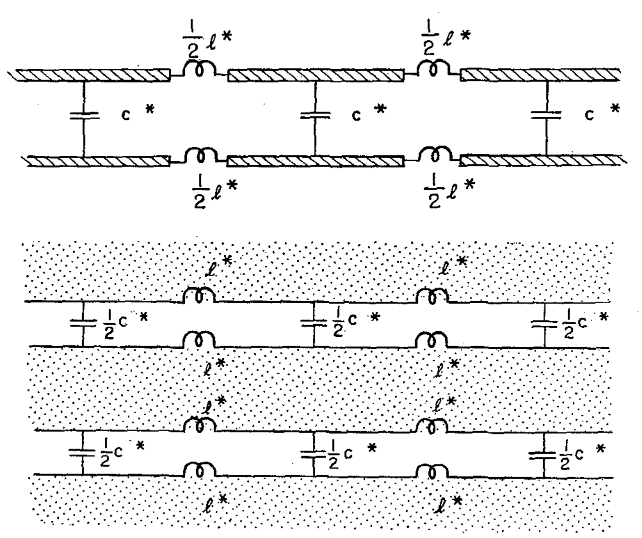


Fig. 6—Equivalent circuits of a two-slot line; diagonal shading indicates magnetic conductors, dots indicate electric conductors.

G. Conclusion

A parallelism between the two-slot line and the two-strip line has been established. Consequently, the well-known solution of the two-wire line equations, as well as complementary measuring techniques, may be applied to the two-slot line. The line parameters that have been derived are true for infinitely thin slots. However, it can be shown[3] that the characteristic impedance Z_{1c} of a very thin two-strip line is given by

$$Z_{1c} = (l/c)^{1/2} = (\mu_0/\epsilon_0)^{1/2}K(k)/K(k'), \qquad (15)$$

where $K(k)$ is the complete elliptic integral of the first kind, $k=a_0/b_0$ is the modulus, $k'^2=1-k^2$, $2a_0$ is the distance between the inner edges of the strips, and $2b_0$ is the distance between the outer edges of the strips. Subject to the condition that the width of the strips is very small compared to the distance between centers so that $a_0 \doteq b_0$, the characteristic impedance is approximately

$$Z_{1c} \doteq (\mu_0/\epsilon_0)^{1/2}\pi^{-1} \ln (4\Delta/\delta), \qquad (16)$$

where $\delta = b_0 - a_0$ is the width of the strip and $\Delta = b_0 + a_0$ is the distance between their centers. From transmission-line theory, the characteristic impedance of a two-wire line of circular conductors is

$$Z_{2c} = (\mu_0/\epsilon_0)^{1/2}\pi^{-1} \ln (b/a) \qquad (17)$$

where b is the distance between the centers of the wires and a is the radius of each. The two strip line evidently behaves like a two-wire line with the same distances between the centers of the conductors and with wires of radius equal to one-quarter the width of the strips.

II. Experimental Study of the Two-Slot Transmission Line

A. The Equipment

The two-slot transmission line is bounded by three pieces of aluminum sheet and an aluminum strip. The ground plane has the over-all dimensions of 6 feet 2 inches by 12 feet 1 inch; it is supported horizontally by a wooden framework at a height about halfway between the floor and the ceiling. The thickness of the aluminum is $\frac{1}{4}$ inch; the center strip is $\frac{1}{2}$ inch by $\frac{1}{4}$ inch in cross section, $7\frac{1}{2}$ feet in length, and supported by a tapered strip of polystyrene that rests on a wooden support.

Several driving devices for the two-slot line were tested. A two-wire line drive [see Fig. 7(a)] was found to be unsatisfactory since the slotted ground plane is an unsymmetrical load that unbalances the two-wire line and causes undesirable radiation. A microstrip drive [see Fig. 7(b)] has the advantage of simplicity in construction and compactness. It consists of a conductor separated from the ground plane by a thin sheet of dielectric. The conductor can be either a flat strip or a wire of small diameter. This conductor is connected to the center-strip of the two-slot line. A coaxial-line drive [see Fig. 7(c)] consists of a piece of coaxial line with its outer conductor deformed into a rectangular shape so that it will fit smoothly onto the ground plane. A $\frac{1}{2}$-inch by $\frac{1}{4}$-inch waveguide was found to be suitable for the outer conductor; a $\frac{3}{16}$-inch diameter brass rod was used as the inner conductor. Two short-circuiting plungers, one on each side of the point of feeding, were provided for matching.

Two different probing systems were employed in the research: the surface-probe system and the enclosed probe system. The surface-probe system consists of a carrier mounted on and movable along a cross-beam,

[3] G. H. Owyang and T. T. Wu, "The approximate parameters of slot lines and their complements," IRE Trans. on Antennas and Propagation, vol. AP-6, pp. 49–55; January, 1958.

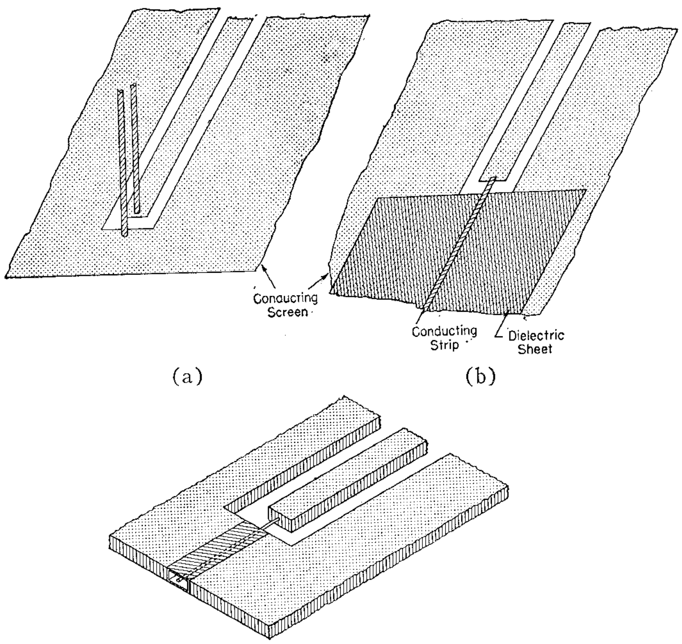


Fig. 7—Methods of driving a two-slot line.

the supporting structure of which rolls on circular steel tracks which are mounted along the sides of the wooden framework. The connection from the probe carrier to the probe is made by a section of stiff transmission line which consists of a piece of a $\frac{1}{4}$-inch O.D. brass tubing slipped over a RG-58 coaxial cable. This brass tubing is threaded and slotted at the upper end to provide a height adjustment.

The enclosed-probe system consists of a movable probe that projects through a slot in a waveguide ($\frac{1}{2}$ inch by $\frac{1}{4}$ inch) which forms the edge of the aluminum sheet. In this system, only the probe itself is exposed to the field to be measured; the connection to the probe and the driving mechanism are either shielded or far away from the point where the measurement is being made. Thus the disturbance caused by the presence of the probing system is minimized and, in addition, the degree of flatness of the metallic sheets has little effect on the signal picked up by the probe.

B. Balancing the Two-Slot Transmission Line

An efficient transmission line should radiate little power. It is well known in two-wire line theory that unbalanced currents radiate. This is also true of a two-slot line in which the currents on the two side plates are unequal at corresponding points. A simple method to determine whether a two-slot line is balanced or not is to record the response of the detector while the probe is moved perpendicularly across the line. The response curve should be symmetrical with respect to the line if a balanced condition is maintained. However, the symmetry of the measured response curve may be affected by the slight variation in the flatness of the ground screen so that an alternative method is desirable.

It is very difficult, if not impossible, to obtain two exactly identical probes; therefore, the direct comparison of the signals picked up by two probes in the two slots will give little information about the condition of balance of the line. With two probes which have slightly different gains, the symmetry of the line current may be determined by the method of cancellation. This is accomplished by adjusting the phases of the signals from the two probes so that the transmission-line modes are opposite in phase while the radiation modes, if they exist, are in phase. Thus a constant resultant signal along the line means that the line is balanced, and the existence of a standing wave in the resultant signal along the line indicates the presence of an unbalanced current in a radiation mode. The probes used are those enclosed in the edges of the two sideplates. These probes are placed at a cross section where the field is strong and are tuned for maximum signal separately by adjusting the tuning stubs. The reading on the variable standard attenuator is recorded. The two circuits are then joined together through a tee with a line stretcher inserted in one probe-circuit. The line stretcher is adjusted for minimum signal and the attenuation of the standard attenuator is reduced to increase the sensitivity of the detector. One of the double-stub tuners may also be adjusted if it helps to decrease the signal. This procedure may be repeated until a true minimum is obtained. The probes are then moved simultaneously along the entire line and the detected signal is noted. Negligible variations in the minimum signal were observed and this minimum signal was more than fifty decibels below (almost noise level) the signal level of either one of the probes. A short piece of lossy cable is inserted in each probe circuit to reduce the possible coupling between the probes.

C. The Transverse Distribution of the Longitudinal Current on a Two-Slot Line

The transverse distribution of the longitudinal current on the metallic surface bounding the two-slot line is measured by moving a surface probe in the direction perpendicular to the slots. The probe is of the shielded-loop type and is oriented with the normal to the plane of the loop parallel to the direction of its movement. Owing to the fact that a loop probe measures the average flux encircled by it, a rectangular loop with round, curved, short sides is used. A loop of such shape has an advantage over a circular loop in being able to measure the average field of a point closer to the metal surface with the same clearance between the probe and the surface. Both the amplitude and the phase of the current are shown in Fig. 8.

The current in the center strip is opposite in phase to that in the side plates and the currents in the two side plates are in phase. The measurements show the current to be concentrated near the edges as expected. The measured apparent decrease in current toward the axis of the center strip is very sensitive to the height of the

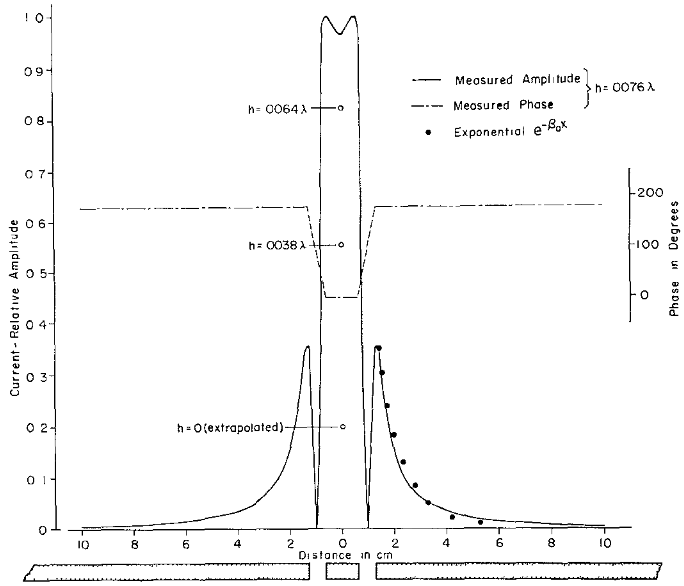


Fig. 8—Distribution of longitudinal current in a two-slot line probe. If the maximum amplitude of the longitudinal current is denoted by I_m, the amplitude of the current along the center line of the center strip by I_c, and the distance between the center of the loop probe to the conducting surface by h, then the observed results are as follows:

h	0.0038λ	0.0064λ	0.0076λ
$\frac{I_c}{I_m}$	0.55	0.82	0.98

At $h=0.0038\ \lambda$, the loop is almost in contact with the conducting surface. If these current ratios are plotted against the distance h and the curve so obtained is extrapolated to the point $h=0$, it is found that the ratio I_c/I_m of the surface current density in the center strip is approximately 0.20. From the distribution of the electromagnetic field it is expected that I_c/I_m has a minimum at the center of the strip and the measurements do verify this fact. Qualitatively, one could imagine the two-slot line to be roughly equivalent to a coplanar four-conductor transmission line. The four conductors are located near the edges of the conducting sheets and the strip. The total currents in the inner two conductors are equal in amplitude and phase; they are equal in amplitude as those in the outer two conductors, but opposite in phase. The current on the side plates decays very rapidly as the distance from the slot increases. This current drops below one-half of one per cent of the peak value within one-quarter of a wavelength from the center of the center strip. It is interesting to note that this decay is almost exponential with distance.

Owing to the nonuniformity in the amplitude of the field configuration, it is not possible to measure the transverse distribution of the transverse current by simply rotating the loop-probe ninety degrees from the position used for measuring the longitudinal current. In this position the loop may respond in its transverse dipole mode to the large E-field in addition to the usual response to the magnetic (or differential electric) field. This was verified by repeating the measurement with a dipole probe with the axis of the dipole perpendicular to the slot; a curve similar to that obtained with the loop probe was obtained.

D. The Measurement of the Attenuation Constant

It is usually very difficult to measure the attenuation constant of a low-loss transmission line. However, if the location of the probe can be measured very accurately along the line, then the method based upon the width of the distribution curve near its minimum is applicable. This method involves the determination of the width Δw_n of the distribution curve at a convenient power level p (usually $p^2=2$ is chosen) above the minimum point at two different locations, w_n and w_{n+m}. The value of the attenuation constant α is given by[4]

$$\alpha = \frac{\beta}{2(p^2-1)^{1/2}} \frac{\Delta w_n - \Delta w_{n+m}}{w_n - w_{n+m}} \text{ nepers per meter.} \quad (18)$$

In the evaluation of the attenuation constant, the portion of the distribution curve near the minimum point is plotted out completely and then extended to locate the minimum. The width of the curve is measured at a power level $p^2=2$ above this minimum. The relative probe position Δw_n is determined by means of two dial-indicators. These dial-indicators are provided with 0.001-inch graduation. The actual location of the point of minimum is not very critical since the value of $(w_n - w_{n+m})$ is of the order of meters.

The measured value of the attenuation constant of the two-slot transmission line is 3.41×10^{-3} nepers per meter which is of the same order of magnitude as that of a two-wire transmission line.

III. Measurement of the Parameters of the Two-Slot Line by the Method of Analogy

A. Introduction

As a substitute for the mathematical analysis of a field problem, the method of field mapping by analogy is useful when the particular field in question is too complicated for rigorous mathematical treatment. It is based upon the correspondence between the steady current field maintained by two oppositely charged electrodes immersed in a homogeneous conducting medium and the electromagnetic field surrounding two similar conductors of infinite length carrying equal and opposite currents.

Since the potential functions ϕ_c in a conductor and ϕ_d in a dielectric both satisfy Laplace's equation, and since the normal components of the electric fields $\boldsymbol{E}_c$ at the boundary between two conductors and $\boldsymbol{E}_d$ at the boundary between two dielectrics satisfy conditions that differ only by a constant factor, it follows that these two cases are analogous. By taking the ratio of the total

[4] King, *op. cit.*, p. 275.

current passing through a volume in a conducting medium, which is bounded by lines of the electric field and two equipotential surfaces at arbitrary points, and the total dielectric flux in a similarly-bounded volume in a dielectric medium, the total capacitance C_d between the equipotential surfaces in the dielectric may be related to the total resistance R_c between the equipotential surfaces in the conducting medium as follows:

$$\frac{\epsilon_d}{\sigma_c} = C_d R_c. \tag{19}$$

It is assumed that the potential differences between the equipotential surfaces in the two cases are equal. ϵ_d is the complex permittivity of the dielectric, σ_c is the conductivity of the conducting medium.

One method of utilizing the analogy between the electric field in a conductor and the electric field in a dielectric is by means of the current distribution in an electrolytic tank filled with a conducting liquid. The electrodes to be investigated are immersed in the liquid and a probe and a bridge-circuit are used to locate the equipotential lines. The orthogonal stream lines are drawn in afterwards to complete the field map. The capacitance C between two electrodes in vacuum may be evaluated from the following formula:

$$C = \epsilon_0 \frac{Q}{V} = \frac{\epsilon_0 \oint_{1c} E_n(s)ds}{\int_{bc} \boldsymbol{E}\cdot d\boldsymbol{s}}, \tag{20}$$

where ϵ_0 is the dielectric constant in vacuum, Q is the total charge on one conductor, V is the potential difference between the conductors, E_n is the normal component of the E-field on the surface of the conductor, $\oint_{1c}$ is the contour-integral taken around the surface of one conductor, and $\int_{bc}$ is the line-integral taken between the two conductors.

The magnitude of the electric field $\boldsymbol{E}$ at any point may be determined from a field plot by drawing a stream line through the point in question, and dividing the potential difference between two equipotential lines lying equal distances from each side of the point by the length of the stream line between them. This method gives good results if the equipotential lines are closely spaced. The normal components of the electric field E_n on the surfaces of the conductor may be obtained by first determining in this manner the value of the E-field along a stream line at several points at different distances from the surface. These values are then plotted against their respective distances from the surface and the curve through them extrapolated to zero distance. Since the electric lines terminate perpendicularly at the conducting surface, the values so obtained are the desired normal components of the electric field. It is usually unnecessary to evaluate the line integral in the denominator of (20), since the potential difference between the electrodes can easily be normalized to unity. Thus, the capacitance between two electrodes may be obtained from the distribution of the field and (20) by numerical integration.

The substitution of (19) into $LC=\mu_0\epsilon_0$ leads to the following relation:

$$L = \mu_0 R_c \sigma_c. \tag{21}$$

Thus, the inductance of two conductors immersed in a dielectric may be obtained from the resistance between the same conductors immersed in another conducting medium.

The attenuation constant of a system of two conductors may be computed from the field distribution in the following manner. If δI is the current carried by an element of surface of width δs on a conductor and of unit length in the direction of propagation, then the total ohmic loss per unit length in both conductors is given by

$$P_L = \frac{R^s}{\zeta_0^2}\oint_{c1} E_n^2 ds + \frac{R^s}{\zeta_0^2}\oint_{c2} E_n^2 ds, \tag{22}$$

where the surface resistance

$$R^s = \left(\frac{\pi f \mu}{\sigma}\right)^{1/2}, \qquad \zeta_0^2 = \frac{\mu_0}{\epsilon_0}$$

is the free-space wave impedance and $\boldsymbol{E}=\zeta_0\boldsymbol{H}$ is used. The contour integrals $\oint_{c1}$ and $\oint_{c2}$ are to be taken around the surfaces of the two conductors, no. 1 and no. 2, respectively. If V is the potential difference between the conductors, then the power transmitted is given by

$$P = VI = \frac{V}{\zeta_0}\oint_{1c} E_n ds. \tag{23}$$

The attenuation constant caused by the ohmic loss in the conductors is, therefore, given by

$$\alpha = \frac{1}{2}\frac{P_L}{P} = \frac{R^s}{2V}\,\frac{\oint_{c1} E_n^2 ds + \oint_{c2} \cdot E_n^2 ds}{\zeta_0 \oint_{1c} E_n ds}. \tag{24}$$

Thus, the attenuation constant is expressed in a form which can be evaluated from the distribution of the field in the conductors. The integrals involved are similar to those in (20) and (21).

B. Measurements in the Electrolytic Tank

The analogous electromagnetic field of the two-slot line was measured in the Harvard Electrolytic Tank, which has been described in detail.[5]

In order to determine the field of the two-slot line with the electrolytic tank, it was necessary to construct a model electrode that had the same cross-sectional view as the two-slot line. Since there is no current crossing the vertical plane of symmetry of the structure, either the

[5] P. A. Kennedy and G. Kent, "The Electrolytic Tank," Harvard University, Cambridge, Mass., Cruft Lab. Tech. Rept. No. 214; 1956.

right or the left half may be removed if an insulating wall is placed along the vertical plane of symmetry. For the same reason, the lower half of the transmission line may be omitted when an insulating wall is placed along the horizontal plane of symmetry, that is, in the slot at a distance one-half the thickness of the conductor from the surface. Thus, only one-fourth of the cross section of the actual two-slot line is required. The model [see Fig. 9(d)] was used to obtain the distribution of the H'-field of a two-slot line.

It was also desired to obtain the distribution of the E-field of the complementary two-strip line. This could be constructed from the distribution of the H-field of a model which had the same cross-sectional view as the actual two-strip line [see Fig. 9(a)]. However, the distribution of the E-field could also be obtained directly from the electrolytic tank by using conjugate electrodes.

The conjugate electrodes are obtained from the original electrodes [see Fig. 9(a)] by using insulators in place of conductors.[6] These are joined together by a thin insulating wall along the horizontal plane of symmetry [see Fig. 9(b)]. A thin conducting surface is placed on each side of this insulating wall where the excitation is applied. Evidently the lines of the current maintained by the conjugate electrodes are orthogonal to those of the original electrodes

It can be shown that the magnetic field $\boldsymbol{H}_1$ maintained with two conducting electrodes immersed in an electrolyte and the electric field $\boldsymbol{E}_2$ of the conjugate electrodes immersed in the same electrolyte satisfy the same field equation and boundary conditions. Therefore, these two fields are analogous to each other and, consequently, the conjugate electrodes may be used to obtain the distribution of the conjugate field of the original electrodes in the electrolytic tank.

By symmetry, the right half of the conjugate electrode [see Fig. 9(b)] may be removed if an insulating wall is erected along the vertical plane of symmetry. Similarly, the lower half may be removed if a conducting surface is placed at the horizontal plane of symmetry. A conducting surface is required here because the stream lines are normal to this plane. The conjugate model is reduced to its final form as shown in Fig. 9(c).

It is interesting to note that the two models, one for measuring the complementary H'-field of the two-slot line [Fig. 9(d)] and the other for measuring the electric field of the two-strip line [Fig. 9(c)], differ only to the extent in which the insulator protrudes out of the conducting surface. In the cases when the conductors are infinitely thin, these two models become identical. Therefore, the same model may be used to measure either the distribution of H'-field of a two-slot line or the distribution of the E-field of a complementary two-strip line by using different insulating inserts.

[6] E. Weber, "Electromagnetic Fields," John Wiley and Sons, Inc., New York, N. Y., vol. 1, p. 186; 1950.

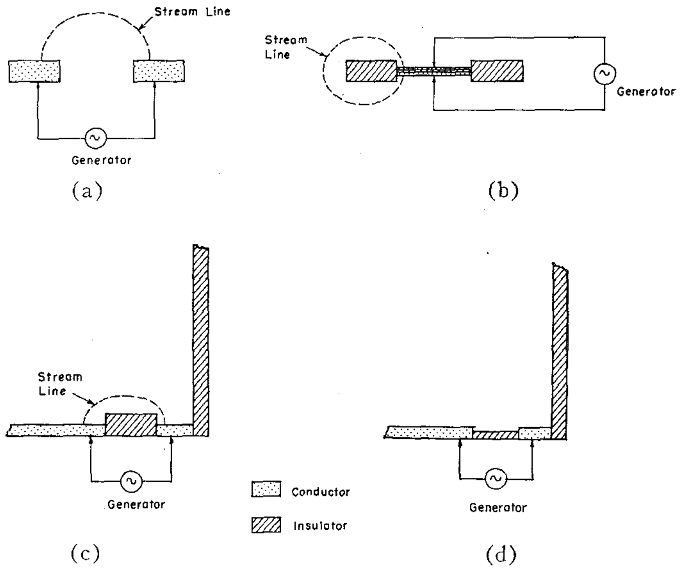


Fig. 9. Two-strip line model and its conjugate model.

In carrying out the measurements in the electrolytic tank, the equipotential lines are plotted directly and the stream lines are then drawn in. An easy way of constructing the orthogonal curves is to construct auxiliary circles[7] (see Fig. 10) between the equipotential lines first, and then to draw curves tangent to those circles and perpendicular to the equipotential lines. Circle-templates are found to be very helpful for this purpose and a reasonably good curvilinear graph usually may be obtained the second trial. A typical example of such a graph is shown in Fig. 10.

The distribution of the field around the two-slot line was obtained by the method mentioned above. The normal component of the electric field at the surface of the electrode was evaluated according to the method described in Section III-A. The capacitance per unit length, the inductance per unit length and the attenuation constant of the two-slot line were computed from (20), (21), and (24) by numerical integrations. These values are listed in Table I.

The capacitance per unit length and the inductance per unit length of the two-slot line were also determined by measuring the resistance between the corresponding electrodes [see (19) and (20)].

The conductivity of the electrolyte may be determined from the measured resistance between the inner and the outer conductor of a model of a coaxial line filled with a known quantity of electrolyte. The leakage conductance per unit length g of a coaxial cable is given by[8]

$$g = \frac{2\pi\sigma}{\ln \dfrac{a_2}{a_1}}, \tag{25}$$

[7] John F. H. Douglass, "Electric, Magnetic, and Thermal Field," vol. 1; and "Experimental Graphical Methods: Mapping," published by the author, ch. 3, p. 3–1, 1953.

[8] King, *op. cit.*, p. 22.

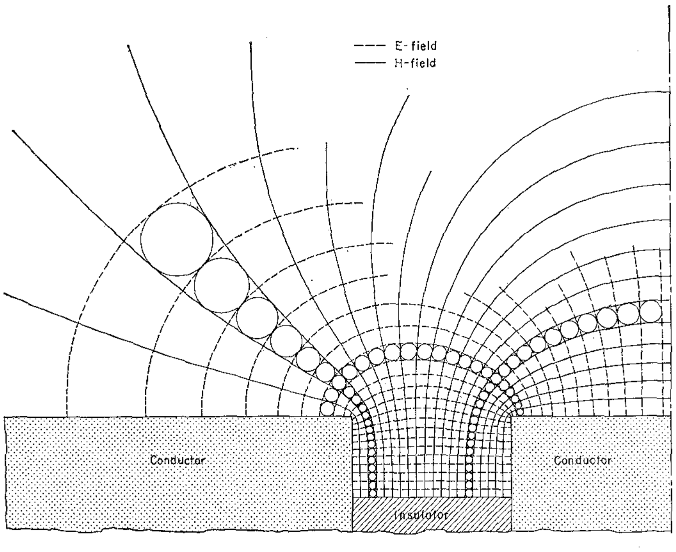


Fig. 10—A typical distribution of the field around a two-slot line.

TABLE I
LINE CONSTANTS OF TWO-SLOT LINE

	α 10^{-3} nepers/m	c $\mu\mu$f/m	l μh/m
Thin Metal Model:			
Theoretical—Analogy from King, "Transmission-line Theory"	—	20.2	0.55
Theoretical—Wu and Owyang*	1.28	27.75	0.402
Electrolytic Tank—Flux Plot	—	26.7	0.416
Electrolytic Tank—Resistance Measurement	—	29.8	0.373
Thick Metal Model:			
Theoretical—Corrected for Thickness (King)†	—	37.9	0.293
Theoretical—Corrected for Thickness (Wu and Owyang*)	—	44.45	0.246
Measurement at 750 mc	3.14	—	—
Electrolytic Tank—Flux Plot	4.21	49.4	0.227
Electrolytic Tank—Resistance Measurement	—	53.5	0.208

* T. T. Wu, and G. H. Owyang "The approximate parameters of slot lines and their complements, IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-6, pp. 49–55; January, 1958.
† The correction for the width of the slot is not included.

where a_2 and a_1 are the radii of the outer and the inner conductors, respectively, and σ is the conductivity of the material between these conductors. By rearranging (25) and multiplying the numerator and the denominator by the factor $(a_2^2-a_1^2)$, the conductivity σ of the medium can be expressed as

$$\sigma = \frac{1}{RV_0}(a_2^2 - a_1^2)\ln\frac{a_2}{a_1}, \tag{26}$$

where R is the resistance between the inner and the outer conductors and V_0 is the volume occupied by the medium between these conductors. The volume V_0 is introduced in (26) because it is easier to fill the coaxial-line model with a definite amount of electrolyte than to measure the depth of the liquid inside the model.

With the conductiviy of the electrolyte and the resistance between the electrodes of the two-slot line model determined, the capacitance per unit length and the inductance per unit length may be computed from (19) and (21). These values are listed in Table I.

C. Conclusion

Both the theoretical and the experimental values of the two-slot line parameters are tabulated in Table I. Despite the fact that none of the conditions required by the theoretical analysis is fulfilled exactly by the actual model under consideration, the two sets of values are not too far apart. The discrepancies are caused both by the degree in which the ideal theoretical model is approximated and by the experimental errors. In the theoretical analysis it is required that the width of the slot be very small compared with the separation and that the metal sheet be relatively thin. The actual line has a separation of only three times the width of the slot which is the same as the thickness of the metal sheet.

The theoretical capacitance per unit length for a thin conducting sheet may be corrected for the case of a thick plate, since the total capacitance consists of the contribution from the top and the bottom surface of the conductor and from the surfaces inside the slots. This involves adding twice the capacitance of two parallel surfaces of unit length with a width equal to the thickness of the plate and a separation equal to the width of the slot. It follows from (20) and (21) that the corrected inductance per unit length may be obtained by multiplying the theoretical value by the ratio of the capacitance due to the surfaces inside the slots and the total capacitance. These values are also tabulated in Table I.

In the evaluation of the line parameters by the method of field mapping, errors may be introduced in the process of measurement, construction of orthogonal curves, evaluation of the E-field (this involves both graphical errors and errors in the approximation), extrapolation of the curves, and the numerical integration of the formulas. If only 1.5 per cent of error is introduced in each of the above possible sources, 9 per cent of error is possible in the final result. By using a larger model, a larger tank, a larger map, and a greater amount of labor, the accuracy of this method may be improved.

The distribution of the electric field about a two-strip line has also been measured by means of conjugate electrodes. This was found to coincide with the distribution of the H-field of the physical dual at distances that are greater than approximately one-half the width of the conductor away from the strip. This indicates that the principle of complementarity[9,10] may be applicable without appreciable error even when the metallic screen has a finite thickness, if a small region near the strip or the complementary slot is excluded.

[9] H. G. Booker, "Slot aerials and their relation to complementary wire aerials—Babinet's principle," *J.I.E.E. (London)*, vol. 93, pt. III A, no. 4, pp. 620–625; 1946.
[10] S. Uda and Y. Mushiake, "The input impedances of slit antennas," "The Technology Reports of the Tohoku University," vol. 14, no. 1, p. 46; 1949.